Highest weight representations of the quantum algebra $\mathbf{U}_h(\mathbf{gl}_\infty)$

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A class of highest weight irreducible representations of the quantum algebra $U_h(gl_\infty)$ is constructed. Within each module a basis is introduced and the transformation relations of the basis under the action of the Chevalley generators are explicitly written.

I. INTRODUCTION

The Lie algebra gl_∞ and its completion and central extension a_∞ [1,2] play an important role in several branches of mathematics and physics. These algebras are of interest as examples of Kac-Moody Lie algebras of infinite type [1,3–5]. They have applications in the theory of nonlinear equations[6], string theory, two-dimensional statistical models[7]. One of us (T.P.) studies new quantum statistics, based on the above algebras (see *Example 2* in Ref. 8 and the references therein), leading to local currents[9] for what are called $A-$spinor fields. It is natural to expect that the deformations of these algebras and their representations may prove useful too.

The quantum analogues of gl_∞ and a_∞ in the sense of Drinfeld[10], namely $U_h(gl_\infty)$ and $U_h(a_\infty)$, were worked out by Levendorskii and Soibelman[11]. These authors have constructed a class of highest weight irreducible representations, writing down explicit expressions for the transformations of the basis under the action of the algebra generators.

In the present note we announce results on certain highest weight irreducible representations (irreps) of $U_h(gl_\infty)$. The $U_h(gl_\infty)$-modules, we study, are labelled by all possible complex sequences (see the end of the introduction for the notation)

$$\{M\} \equiv \{M_i\}_{i\in \mathbf{Z}} \in \mathbf{C}^\infty, \;\; \text{such that} \;\; M_i - M_j \in \mathbf{Z}_+, \quad \forall i < j. \tag{1}$$

The signatures of the $U_h(gl_\infty)$-modules of Levendorskii and Soibelman[11] consist of all those sequences $\{M^{(s)}\}$, $s \in \mathbf{Z}$, from (1), for which $M_i^{(s)} = 1$, if $i < s$ and $M_i^{(s)} = 0$ for $i \geq s$.

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Here we use essentially the results of Ref. 12, where a class of highest weight irreps of gl_∞ was obtained. The corresponding gl_∞-modules are labelled with the same signatures (1). We show in fact that each $U_h(gl_\infty)$-module with a signature $\{M\}$ is an appropriate deformation of a gl_∞-module with the same signature. Our results are based on a proof of a conjecture we did: the replacement of (almost) all number multiples in the transformation relations of the basis $\Gamma(\{M\})$ with formal power series according to

$$x \to [x] = \frac{q^x - q^{-x}}{q - q^{-1}} \in \mathbf{C}[[h]], \quad q = e^{h/2} \tag{2}$$

turns the gl_∞-module with a signature $\{M\}$ into a $U_h(gl_\infty)$-module with the same signature.

It is certainly not surprising such a conjecture holds, because it takes place for all finite-dimensional irreps of $U_h(gl(n))$[13] and of the superalgebras $U_h(gl(n/1))$, $n \in \mathbf{N}$[14], if they are written in a Gel'fand-Zetlin basis, and also for all essentially typical representations of the superalgebras $U_h(gl(n/m))$[15]. On the other hand counter examples are also available[16]. We see our main contribution to be in the nontrivial proof of the conjecture, not in its formulation.

Throughout the paper we use the notation (most of them standard):

$\mathbf{N}$ - all positive integers;

$\mathbf{Z}_+$ - all non-negative integers;

$\mathbf{Z}$ - all integers;

$\mathbf{Q}$ - all rational numbers;

$\mathbf{C}$ - all complex numbers;

$\mathbf{C}[[h]]$ - the ring of all formal power series in h over $\mathbf{C}$;

$\Gamma(\{M\})$ - the C-basis of a module with a signature $\{M\}$;

$q = e^{h/2}$.

II. THE ALGEBRA $\mathbf{U}_h(\mathbf{gl}_\infty)$

Following Ref. 11 we recall the definition of $U_h(gl_\infty)$. It is the Hopf algebra, which is topologically free module over $\mathbf{C}[[h]]$ (complete in h-adic topology), with generators $\{E_i, F_i, H_i\}_{i\in\mathbf{Z}}$, the Chevalley generators, and

1. Cartan relations:

$$\begin{aligned}
&[H_i, H_j] = 0,\\
&[H_i, E_j] = (\delta_{ij} - \delta_{i,j+1})E_j,\\
&[H_i, F_j] = -(\delta_{ij} - \delta_{i,j+1})F_j,\\
&[E_i, F_j] = \delta_{ij}\frac{q^{H_i - H_{i+1}} - q^{-H_i + H_{i+1}}}{q - q^{-1}}.
\end{aligned} \tag{3}$$

2. E-Serre relations:

$$\begin{aligned}
&E_iE_j = E_jE_i \quad \text{if} \quad |i-j| \neq 1,\\
&E_i^2E_{i+1} - (q+q^{-1})E_iE_{i+1}E_i + E_{i+1}E_i^2 = 0,\\
&E_{i+1}^2E_i - (q+q^{-1})E_{i+1}E_iE_{i+1} + E_iE_{i+1}^2 = 0.
\end{aligned} \tag{4}$$

3. F-Serre relations:

$$\begin{aligned}
&F_iF_j = F_jF_i \quad \text{if} \;\; |i-j| \neq 1,\\
&F_i^2F_{i+1} - (q+q^{-1})F_iF_{i+1}F_i + F_{i+1}F_i^2 = 0,\\
&F_{i+1}^2F_i - (q+q^{-1})F_{i+1}F_iF_{i+1} + F_iF_{i+1}^2 = 0.
\end{aligned} \tag{5}$$

A counity ε, a comultiplication Δ and an antipode S are defined as:

$$\begin{aligned}
&\varepsilon(E_i) = \varepsilon(F_i) = \varepsilon(H_i) = 0,\\
&\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i,\\
&\Delta(E_i) = E_i \otimes q^{(H_i-H_{i+1})/2} + q^{(-H_i+H_{i+1})/2} \otimes E_i,\\
&\Delta(F_i) = F_i \otimes q^{(H_i-H_{i+1})/2} + q^{(-H_i+H_{i+1})/2} \otimes F_i,\\
&S(H_i) = -H_i, \quad S(E_i) = -qE_i, \quad S(F_i) = -q^{-1}F_i.
\end{aligned} \tag{6}$$

Throughout the tensor products are topological, namely the algebraic tensor products are replaced with their completion in the h-adic topology.

Note that $\{E_i, F_i, H_i\}_{i\in \mathbf{N}}$ generate a Hopf subalgebra $U_h(gl_0(\infty))$ of $U_h(gl_\infty)$.

III. REPRESENTATIONS OF $\mathbf{U}_h(\mathbf{gl}_\infty)$

Let W be a topologically free $\mathbf{C}[[h]]$-module. We recall that a $\mathbf{C}[[h]]$-homomorphism $\rho : U_h(gl_\infty) \to End\, W$ is a representation of $U_h(gl_\infty)$ in W (equivalently, W is a $U_h(gl_\infty)$-module) provided ρ is continuous in the h-adic topology.

We proceed to define the $U_h(gl_\infty)$-module $V(\{M\})$ with a highest weight $\{M\}$ and its (topological) basis. The basis $\Gamma(\{M\})$, called a central basis (C-bases), is formally the same as the one introduced in Ref. 12 for description of the representations of gl_∞. It consists of all $C-$ patterns

$$(M) \equiv \left[\begin{array}{cccccccc}
.., & M_{1-\theta-k}, & \ldots, & M_{-1}, & M_0, & M_1, & \ldots & M_{k+\theta-1}, ...\\
.., & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
 & M_{1-\theta-k,2k+\theta-1}, & \ldots, & M_{-1,2k+\theta-1}, & M_{0,2k+\theta-1}, & M_{1,2k+\theta-1}, & \ldots & M_{k+\theta-1,2k+\theta-1}\\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & & M_{-1,3}, & M_{03}, & M_{13} & & \\
 & & & M_{-1,2}, & M_{02} & & & \\
 & & & & M_{01} & & &
\end{array}\right], \tag{7}$$

where $k \in \mathbf{N},\ \theta = 0, 1$. Each such pattern is an ordered collection of complex numbers

$$M_{i,2k+\theta-1}, \quad \forall k \in \mathbf{N}, \quad \theta = 0, 1, \qquad i \in [-\theta-k+1, k-1] \equiv \{-\theta-k+1, -\theta-k+2, \ldots, k-1\}, \tag{8}$$

which satisfy the conditions:

(i) there exist $N((M)) \in \mathbf{N}$ such that

$$M_{i,2k+\theta-1} = M_i, \;\; \forall k > N((M)), \quad \theta = 0, 1, \;\; i \in [-\theta-k+1, k-1]; \tag{9}$$

(ii)

$$M_{i+\theta-1,2k+\theta}-M_{i,2k+\theta-1}\in \mathbf{Z}_+,\ M_{i,2k+\theta-1}-M_{i+\theta,2k+\theta}\in \mathbf{Z}_+,\ \forall k\in \mathbf{N},\ \theta=0,1,\ i\in[1-\theta-k,k-1]. \quad (10)$$

Denote by $V_0(\{M\})$ the free $\mathbf{C}[[h]]$-module with generators $\Gamma(\{M\})$ and let $V(\{M\})$ be its completion in the h-adic topology. $V(\{M\})$ is topologically free $\mathbf{C}[[h]]$-module with (topological) basis $\Gamma(\{M\})$. In order to give somewhat more explicit description of $V(\{M\})$ set V to be the (formal) complex linear envelope of $\Gamma(\{M\})$. Then $V_0(\{M\})=V\otimes_{\mathbf{C}}\mathbf{C}[[h]]$ and $V(\{M\})$ consists of all formal power series in h with coefficients in V:

$$v=v_0+v_1h+v_2h^2+\dots \quad (v_0,v_1,v_2,\dots\in V). \quad (11)$$

$End\,V(\{M\})$ is a $\mathbf{C}[[h]]$-module, consisting of all $\mathbf{C}[[h]]-$linear maps of $V(\{M\})$. If $a\in End\,V(\{M\})$, then

$$av=av_0+(av_1)h+(av_2)h^2+\dots. \quad (12)$$

Therefore the transformation of $V(\{M\})$ under the action of $a\in End\,V(\{M\})$ is completely defined, if a is defined on $\Gamma(\{M\})$.

We pass to turn $V(\{M\})$ into a $U_h(gl_\infty)$ module. To this end introduce first some appropriate notation[12]. Denote by $(M)_{\pm\{j,p\}}$ and $(M)^{\pm\{l,q\}}_{\pm\{j,p\}}$ the patterns obtained from the C-pattern (M) (7) after the replacements

$$M_{lq}\to M_{lq}\pm 1,\quad M_{jp}\to M_{jp}\pm 1, \quad (13)$$

correspondingly, and let

$$S(j,l;\nu)=\begin{cases}(-1)^\nu & \text{for } j=l\\ 1 & \text{for } j<l\\ -1 & \text{for } j>l\end{cases},\quad \theta(i)=\begin{cases}1 & \text{for } i\geq 0\\ 0 & \text{for } i<0\end{cases},\quad L_{ij}=M_{ij}-i. \quad (14)$$

Set moreover

$$E_i^0=F_i,\quad E_i^1=E_i,\quad i\in\mathbf{Z}. \quad (15)$$

Let $\{\rho(E_i),\rho(F_i),\rho(H_i)\}_{i\in\mathbf{Z}}$ be a collection of $\mathbf{C}[[h]]$-endomorphisms of $V(\{M\})$, defined on any C-pattern $(M)\in\Gamma(\{M\})$, as follows:

$$\rho(E_{-1}^{1-\mu})(M)=([L_{-1,2}-L_{0,1}-\mu][L_{0,1}-L_{0,2}+\mu])^{1/2}(M)_{-(-1)^\mu\{0,1\}},\qquad \mu=0,1\ , \quad (16)$$

$$\begin{aligned}\rho(E^{\mu}_{(-1)^\nu i-1})(M)=&-\sum_{j=1-i-\nu}^{i-1}\ \sum_{l=-i}^{i+\nu-1}S(j,l;\nu)\\ &\times\Bigg(-\frac{\prod_{k\neq l=-i}^{i+\nu-1}[L_{k,2i+\nu}-L_{j,2i+\nu-1}-(-1)^\nu\mu]\prod_{k=1-i}^{i+\nu-2}[L_{k,2i+\nu-2}-L_{j,2i+\nu-1}-(-1)^\nu\mu]}{\prod_{k\neq j=1-i-\nu}^{i-1}[L_{k,2i+\nu-1}-L_{j,2i+\nu-1}][L_{k,2i+\nu-1}-L_{j,2i+\nu-1}+(-1)^{\mu+\nu}]}\\ &\times\frac{\prod_{k=-i-\nu}^{i}[L_{k,2i+\nu+1}-L_{l,2i+\nu}+(-1)^\nu(1-\mu)]\prod_{k\neq j=1-i-\nu}^{i-1}[L_{k,2i+\nu-1}-L_{l,2i+\nu}+(-1)^\nu(1-\mu)]}{\prod_{k\neq l=-i}^{i+\nu-1}[L_{k,2i+\nu}-L_{l,2i+\nu}][L_{k,2i+\nu}-L_{l,2i+\nu}+(-1)^{\mu+\nu}]}\Bigg)^{1/2}\\ &\times(M)^{-(-1)^{\mu+\nu}\{l,2i+\nu\}}_{-(-1)^{\mu+\nu}\{j,2i-1+\nu\}},\quad i\in\mathbf{N},\quad \mu,\nu=0,1\ ,\end{aligned} \quad (17)$$

$$\rho(H_i)(M)=\left(\sum_{j=-|i|}^{|i|+\theta(i)-1}M_{j,2|i|+\theta(i)}-\sum_{j=-|i|+1-\theta(i)}^{|i|-1}M_{j,2|i|+\theta(i)-1}\right)(M),\quad i\in\mathbf{Z}. \quad (18)$$

If a pattern from the right hand side of (17) does not belong to $\Gamma(\{M\})$, i.e., it is not a C-pattern, then the corresponding term has to be deleted. (The coefficients in front of all such patterns are undefined, they contain zero multiples in the denominators. Therefore an equivalent statement is that all terms with zeros in the denominators have to be removed). With this convention all coefficients in front of the $C-$patterns in r.h.s of (16)-(18) are well defined as elements from $\mathbf{C}[[h]]$.

Proposition 1. The endomorphisms $\{\rho(E_i), \rho(F_i), \rho(H_i)\}_{i\in\mathbf{Z}}$ satisfy Eqs. (3)-(5) with $\rho(E_i), \rho(F_i), \rho(H_i)$ substituted for E_i, F_i, H_i, respectively.

In the nondeformed case ($h \to 0$) the above transformation relations define a representation of gl_∞, if throughout in (16) and (17) the "quantum" brackets [ ] are assumed to be usual brackets. The nondeformed Eqs. (16)-(18) were derived in Ref. 12 from the $gl_0(\infty)$ transformation relations of the Gel'fand-Zetlin basis. The derivation was based on an isomorphism φ of gl_∞ onto $gl_0(\infty)$, defined as

$$\varphi(E_{ij}) = E_{2|i|+\theta(i),2|j|+\theta(j)}, \quad i, j \in \mathbf{Z}, \tag{19}$$

where $\{E_{ij}\}_{i,j\in\mathbf{Z}}$ and $\{E_{ij}\}_{i,j\in\mathbf{N}}$ are the Weyl generators of gl_∞ and $gl_0(\infty)$, respectively. In the deformed case ($h \neq 0$) φ does not define any more an isomorphism of $U_h(gl_\infty)$ onto $U_h(gl_0(\infty))$. In fact we do not know whether $U_h(gl_\infty)$ is isomorphic to its subalgebra $U_h(gl_0(\infty))$. The proof of *Proposition 1* is based on a direct verification of the relations (3)-(5). This verification is extremely lengthy and in certain cases highly nontrivial. The most difficult to check are the last Cartan relations in (3), corresponding to $i = j$. In order to show they hold, one has to prove as an intermediate step that the following identities are fulfilled:

$$\sum_{j=1-k}^{k-1}\sum_{l=-k}^{k-1} \frac{\prod_{i\neq l=-k}^{k-1}[L_{i,2k}-L_{j,2k-1}-1]\prod_{i=1-k}^{k-2}[L_{i,2k-2}-L_{j,2k-1}-1]\prod_{i=-k}^{k}[L_{i,2k+1}-L_{l,2k}]\prod_{i\neq j=1-k}^{k-1}[L_{i,2k-1}-L_{l,2k}]}{\prod_{i\neq j=1-k}^{k-1}[L_{i,2k-1}-L_{j,2k-1}][L_{i,2k-1}-L_{j,2k-1}-1]\prod_{i\neq l=-k}^{k-1}[L_{i,2k}-L_{l,2k}][L_{i,2k}-L_{l,2k}-1]}$$

$$-\sum_{j=1-k}^{k-1}\sum_{l=-k}^{k-1} \frac{\prod_{i\neq l=-k}^{k-1}[L_{i,2k}-L_{j,2k-1}]\prod_{i=1-k}^{k-2}[L_{i,2k-2}-L_{j,2k-1}]\prod_{i=-k}^{k}[L_{i,2k+1}-L_{l,2k}+1]\prod_{i\neq j=1-k}^{k-1}[L_{i,2k-1}-L_{l,2k}+1]}{\prod_{i\neq j=1-k}^{k-1}[L_{i,2k-1}-L_{j,2k-1}][L_{i,2k-1}-L_{j,2k-1}+1]\prod_{i\neq l=-k}^{k-1}[L_{i,2k}-L_{l,2k}][L_{i,2k}-L_{l,2k}+1]}$$

$$=\Big[\sum_{j=-k+1}^{k-1}L_{j,2k-1}-\sum_{j=-k+1}^{k-2}L_{j,2k-2}-\sum_{j=-k}^{k}L_{j,2k+1}+\sum_{j=-k}^{k-1}L_{j,2k}-1\Big], \quad k\in\mathbf{N}, \tag{20}$$

$$\sum_{j=-k}^{k-1}\sum_{l=-k}^{k} \frac{\prod_{i\neq l=-k}^{k}[L_{i,2k+1}-L_{j,2k}+1]\prod_{i=1-k}^{k-1}[L_{i,2k-1}-L_{j,2k}+1]\prod_{i=-k-1}^{k}[L_{i,2k+2}-L_{l,2k+1}]\prod_{i\neq j=-k}^{k-1}[L_{i,2k}-L_{l,2k+1}]}{\prod_{i\neq j=-k}^{k-1}[L_{i,2k}-L_{j,2k}][L_{i,2k}-L_{j,2k}+1]\prod_{i\neq l=-k}^{k}[L_{i,2k+1}-L_{l,2k+1}][L_{i,2k+1}-L_{l,2k+1}+1]}$$

$$-\sum_{j=-k}^{k-1}\sum_{l=-k}^{k} \frac{\prod_{i\neq l=-k}^{k}[L_{i,2k+1}-L_{j,2k}]\prod_{i=1-k}^{k-1}[L_{i,2k-1}-L_{j,2k}]\prod_{i=-k-1}^{k}[L_{i,2k+2}-L_{l,2k+1}-1]\prod_{i\neq j=-k}^{k-1}[L_{i,2k}-L_{l,2k+1}-1]}{\prod_{i\neq j=-k}^{k-1}[L_{i,2k}-L_{j,2k}][L_{i,2k}-L_{j,2k}-1]\prod_{i\neq l=-k}^{k}[L_{i,2k+1}-L_{l,2k+1}][L_{i,2k+1}-L_{l,2k+1}-1]}$$

$$=\Big[\sum_{j=-k-1}^{k}L_{j,2k+2}-\sum_{j=-k}^{k}L_{j,2k+1}-\sum_{j=-k}^{k-1}L_{j,2k}+\sum_{j=-k+1}^{k-1}L_{j,2k-1}-1\Big], \quad k\in\mathbf{N}. \tag{21}$$

We prove these identities (and several others, which are simpler) in a paper to come, where we extend our results to the completion and central extension $U_h(a_\infty)$ of $U_h(gl_\infty)$.

A (topological) basis $\{e_n\}_{n\in\mathbf{N}}$ in $U_h(gl_\infty)$ was given in Ref. 11. Each basis vector e_n is an appropriate $\mathbf{C}[[h]]$-polynomial in the Chevalley generators. The $\mathbf{C}[[h]]$-span $W[[h]]$ of the basis is dense in $U_h(gl_\infty)$. It consists of all $\mathbf{C}[[h]]$-polynomials in the Chevalley generators. Extend the domain of definition of ρ on $W[[h]]$ in a natural way: if ρ has already been defined on $a, b \in W[[h]]$, then set

$$\rho(\alpha a+\beta b)=\alpha\rho(a)+\beta\rho(b),\quad \rho(ab)=\rho(a)\rho(b),\quad a,b\in W[[h]],\quad \alpha,\beta\in\mathbf{C}[[h]]. \tag{22}$$

Denote by $U(gl_\infty)$ the $\mathbf{C}-$linear span of the basis. Clearly $U(gl_\infty)\subset W[[h]]$ and therefore ρ is defined on $U(gl_\infty)$. $U_h(gl_\infty)$ consists of all elements of the form

$$a=\sum_{i=0}^{\infty}a_ih^i,\quad (a_0,a_1,a_2,\ldots\in U(gl_\infty)). \tag{23}$$

For any i, according to (11),

$$\rho(a_i)=\sum_{j=0}^{\infty}\alpha_{ij}h^j\in V(\{M\}),\quad \alpha_{ij}\in V. \tag{24}$$

Therefore,

$$\sum_{i=0}^{\infty}\rho(a_i)h^i=\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}\alpha_{ij}h^j\right)h^i=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\alpha_{n-m,m}\right)h^n\in V(\{M\}). \tag{25}$$

Using (25), extend ρ on $U_h(gl_\infty)$:

$$\rho(a)=\sum_{i=0}^{\infty}\rho(a_i)h^i\in V(\{M\})\quad \forall\ a\in U_h(gl_\infty). \tag{26}$$

Hence ρ is a well defined map from $U_h(gl_\infty)$ into $End\,V(\{M\})$,

$$\rho:\ U_h(gl_\infty)\to End\ V(\{M\}). \tag{27}$$

Let $a\in U_h(gl_\infty)$. Then any neighbourhood $W(\rho(a))$ of $\rho(a)$ contains a basic neighbourhood $\rho(a)+h^nEnd\ V(\{M\})\subset W(\rho(a))$. We have in mind the h-adic topology both in $U_h(gl_\infty)$ and $End\ V(\{M\})$. Evidently

$$\rho\left(a+h^nU_h(gl_\infty)\right)\subset\rho(a)+h^nEnd\,V(\{M\})\subset W(\rho(a))\ \ \forall a\in U_h(gl_\infty). \tag{28}$$

Therefore ρ is a continuous map. It satisfies Eq. (22) for any $a,b\in U_h(gl_\infty)$ and $\alpha,\beta\in\mathbf{C}[[h]]$. Therefore ρ is a $\mathbf{C}[[h]]$-homomorphism of $U_h(gl_\infty)$ in $End\,V(\{M\})$. We have obtained the following result.

Proposition 2. The map (27), acting on the $C-$basis according to Eqs. (16)-(18), defines a representation of $U_h(gl_\infty)$ in $V(\{M\})$.

Any $U_h(gl_\infty)$-module $V(\{M\})$ is a highest weight module with respect to the "Borel" subalgebra N_+, consisting of all $\mathbf{C}[[h]]$-polynomials of the unity and $\{E_i\}_{i\in\mathbf{Z}}$. The highest weight vector $(\hat{M})$, which by definition satisfies the condition $\rho(N_+)(\hat{M})=0$, corresponds to the one from (7) with

$$\hat{M}_{i,2k+\theta-1}=M_i,\quad \forall k\in\mathbf{N},\quad \theta=0,1,\qquad i\in[-\theta-k+1,k-1]. \tag{29}$$

Moreover, $V(\{M\}) = \rho(U_h(gl_\infty))(\hat{M})$. Since $V(\{M\})$ contains no other singular vectors, vectors annihilated from $\rho(N_+)$, each $V(\{M\})$ is an irreducible $U_h(gl_\infty)$-module. The proof of the latter follows from the results in Ref. 12 and the observation that each (deformed) matrix element in the transformation relations (16)-(18) is zero only if the corresponding nondeformed matrix element vanishes.

We note in conclusion that all our results remain valid for $h \notin i\pi\mathbf{Q}$, namely in the case q is a number, which is not a root of 1.

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